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THE CUSP FORMS OF WEIGHT 3 ON $\Gamma_2(2, 4, 8)$

BERT VAN GEEMEN AND DUCO VAN STRATEN

ABSTRACT. The congruence subgroup $\Gamma_2(2, 4, 8)$ of the group Γ_2 of 4×4 integral symplectic matrices is contained in $\Gamma_2(4)$ and contains $\Gamma_2(8)$, with $\Gamma_2(n)$ the principal congruence subgroup of level n. The Satake compactification of the quotient of the three-dimensional Siegel upper half space by $\Gamma_2(2, 4, 8)$ is shown to be a complete intersection of ten quadrics in $\mathbb{P}^{13}$. We determine the space of global holomorphic three forms on this space, which coincides with the space of cusp forms of weight 3 on $\Gamma_2(2, 4, 8)$; it has dimension 2283. Finally, we study the action of the Hecke operators on this space and consider the Andrianov L-functions of some eigenforms.

1. INTRODUCTION

1.1. In this paper we study the cusp forms of weight 3 on the congruence subgroup $\Gamma_g(2, 4, 8)$ of $\Gamma_g := Sp_{2g}(\mathbb{Z})$ in case $g = 2$.

Recall that $\Gamma_g(n)$ consists of the matrices which are $\equiv I \bmod n$, that

$$\Gamma_g(4, 8) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(4) : \ \operatorname{diag}(B) \equiv \operatorname{diag}(C) \equiv 0 \bmod 8 \right\},$$

and in [6] the following (normal) subgroup of Γ_g was defined:

$$\Gamma_g(2, 4, 8) := \left\{ \begin{pmatrix} I + 4A' & B \\ C & I + 4D' \end{pmatrix} \in \Gamma_g(4, 8) : \ \operatorname{trace}(A') \equiv 0 \bmod 2 \right\}.$$

In particular,

$$\Gamma(8) \hookrightarrow \Gamma(2, 4, 8) \hookrightarrow \Gamma(4, 8) \hookrightarrow \Gamma(4).$$

The Siegel upper half plane $\mathbb{H}_g$ is the analytic variety consisting of $g \times g$ complex symmetric matrices with positive definite imaginary part. For a function $f: \mathbb{H} \to \mathbb{C}$, $M \in \Gamma_g$ and $k \in \mathbb{N}$ one defines

$$f|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix} (\tau) = \det(C\tau + D)^{-k} f((A\tau + B)(C\tau + D)^{-1}).$$

Let Γ' be a congruence subgroup of Γ_g, that is, $\Gamma_g(n) \subset \Gamma'$ for some n. A modular form of weight k for Γ' is a holomorphic function f on $\mathbb{H}_g$

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satisfying $f|_k M = f$ for all $M \in \Gamma'$. The $\mathbb{C}$-vector space of such functions is denoted by $M_k(\Gamma')$.

One defines the Siegel operator Ψ, mapping $f \in M_k(\Gamma')$ to a function on $\mathbb{H}_{g-1}$, by

$$\Psi(f)(\tau) := \lim_{t\to\infty} f\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & it\end{smallmatrix}\right)\right), \qquad \tau \in \mathbb{H}_{g-1}.$$

The subspace $S_k(\Gamma')$ of $M_k(\Gamma')$, called the space of cusp forms, is defined by

$$S_k(\Gamma') = \{f \in M_k(\Gamma') \colon \Psi(f|_k M)(\tau) = 0\ \forall \tau \in \mathbb{H}_{g-1},\ \forall M \in \Gamma_g\}.$$

1.2. In case the group Γ' acts without fixed points on $\mathbb{H}_g$ (for example, if $\Gamma' \subset \Gamma_g(n)$ and $n \geq 3$), the space $M_{g+1}(\Gamma')$ corresponds to the space of holomorphic $\frac{1}{2}g(g+1)$-forms on the quasi-projective variety $X^0 = \mathbb{H}_g/\Gamma'$. This correspondence is given by $\omega \mapsto f$ when

$$\pi \colon \mathbb{H}_g \longrightarrow X^0, \qquad \pi^*\omega = f\left(\bigwedge d\tau_{ij}\right).$$

The subspace of those forms which extend to (any) smooth compactification $\widetilde{X}$ of X^0 is exactly $S_{g+1}(\Gamma')$. In particular,

$$S_{g+1}(\Gamma') \cong H^0(\widetilde{X}, \Omega_{\widetilde{X}}^{\frac{1}{2}g(g+1)}).$$

A remarkable aspect of this result is that the 'cusp form condition' need only be checked at points in the boundary of the Satake compactification which are in quotients of $\mathbb{H}_{g-1}$, rather than at all points which are in quotients of $\mathbb{H}_k$ with $0 \leq k \leq g-1$ (this can be generalized to other symmetric domains, see [1, Chapter IV]). We will happily exploit this fact.

1.3. In the case $g = 2$ (where we will omit the subscript g) and $\Gamma' = \Gamma(2, 4, 8)$, the variety X^0 can be described explicitly as a Zariski open subset of a projective variety $X \subset \mathbb{P}^{13}$. The embedding of X^0 into $\mathbb{P}^{13}$ is given by certain theta constants. The variety X is the complete intersection of ten quadrics, which can easily be written down explicitly. Using this, and combinatorics of theta constants, we can determine the space $H^0(\widetilde{X}, \Omega^3_{\widetilde{X}})$, and thus also the space $S_3(\Gamma(2, 4, 8))$. (We use the computer program 'Macaulay' for the manipulations with ideals of polynomials.)

On the space $S_3(\Gamma(2, 4, 8))$ there acts the finite group $\Gamma/\Gamma(2, 4, 8)$, and we determine the decomposition into irreducible subrepresentations.

In the last sections we study the action of the Hecke algebra on $S_3(\Gamma(2, 4, 8))$. The action of this algebra is induced by correspondences. In this case these are codimension 3 cycles on $X^0 \times X^0$ and by 'pullback-push forward' they give linear maps on $S_3(\Gamma(2, 4, 8))$. The definition of these cycles is in terms of isogenies of abelian varieties. Similar to the elliptic modular case, one has a congruence relation which relates the action of the Hecke operators on $S_3(\Gamma(2, 4, 8))$ to the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^3(\widetilde{X}, \mathbb{Q}_l)$. It is therefore of some interest to determine the eigenspaces and eigenvalues of these operators. We determined the Hecke polynomials, which describe the Hecke action, for several cusp forms and for some small primes p.

Most of the forms we consider appear to be obtained via liftings from modular forms on subgroups of $SL_2(\mathbb{Z})$. In one case the Hecke polynomials suggest that the modular form is related to a Hecke character of the field of eight roots of unity (the form g_1). There is one case in which the Hecke polynomials of the cusp form do not allow one of these interpretations (the form g_2). In this paper we do not actually try to prove that most of the forms are indeed liftings.

2. Combinatorics of theta characteristics

2.1. The modular forms we consider are linear combinations of products of theta constants. For $m = \frac{1}{2}(m', m'') \in \mathbb{R}^2 \times \mathbb{R}^2$ with $m'_j, m''_j \in \{0, 1\}$ we define the theta constant $\theta_m : \mathbb{H}_2 \to \mathbb{C}$ with (half-integral) (theta) characteristic m by

$$\theta_m(\tau) := \sum_{k \in \mathbb{Z}^2} \exp(2\pi i[\tfrac{1}{2}(k + \tfrac{m'}{2})\tau^t(k + \tfrac{m'}{2}) + (k + \tfrac{m'}{2})^t(\tfrac{m''}{2})]).$$

The theta constant is not identically zero if and only if the theta characteristic m is *even*, i.e., $m''^t m'' \in 2\mathbb{Z}$. There are ten even theta characteristics. If $m = \frac{1}{2}(a, b, c, d)$ we will also write

$$\theta_m(\tau) = \theta[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}](\tau).$$

Under the action of Γ on $\mathbb{H}_2$ these ten theta null's are permuted (up to a root of unity times a common factor, cf. 5.2). Therefore, Γ acts on the set C_1 of the ten even characteristics. The action of $M \in \Gamma$ is given by (cf. [8, V.6]):

$$M : C_1 \to C_1, \qquad M * m := n, \qquad n = mM^{-1} + \tfrac{1}{2}((C^tD)_0, (A^tB)_0) \bmod 1.$$

Here, $M \in \Gamma$ is the matrix with blocks

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and $(C^tD)_0$, $(A^tD)_0$ are the diagonals of the matrices C^tD and A^tB, respectively, viewed as row vectors.

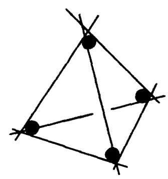
This action of Γ factors over $\Gamma/\Gamma(2) \cong S_6$, the group of permutations of the set $S = \{1, 2, \ldots, 6\}$. The ten even theta characteristics then correspond to the $\frac{1}{2}\binom{6}{3} = 10$ partitions of S into two subsets with three elements each (cf. [9]); such a partition is called a triadic syntheme. The action of S_6 on C_1 is then easy to follow.

2.2. Associated with each $m \in C_1$ there is a quadratic form Q_m in the variables X_0, X_1, X_2, X_3 and a quadric $V_m = V(Q_m)$ in $\mathbb{P}^3 = \mathbb{P}(X_0, X_1, X_2, X_3)$. The Q_m's are defined by the relation (cf. [8, IV.1]):

$$\theta_m^2(\tau) = Q_m(\theta[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}](2\tau), \theta[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}](2\tau), \theta[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}](2\tau), \theta[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}](2\tau)). \tag{2.3}$$

	m	triad	Q_m
1	$\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	156 234	$X_0^2+X_1^2+X_2^2+X_3^2$
2	$\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$	134 256	$X_0^2-X_1^2+X_2^2-X_3^2$
3	$\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$	146 235	$X_0^2+X_1^2-X_2^2-X_3^2$
4	$\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$	135 246	$X_0^2-X_1^2-X_2^2+X_3^2$
5	$\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$	124 356	$2(X_0X_1+X_2X_3)$
6	$\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$	145 236	$2(X_0X_2+X_1X_3)$
7	$\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$	126 345	$2(X_0X_3+X_1X_2)$
8	$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	125 346	$2(X_0X_1-X_2X_3)$
9	$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$	136 245	$2(X_0X_2-X_1X_3)$
10	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	123 456	$2(X_0X_3-X_1X_2)$

The ten quadrics determine an interesting configuration of 30 lines (15 pairs of skew lines) and 60 points (vertices of 15 tetrahedrons). By a tetrahedron we mean the algebraic variety consisting of the union of six lines, the edges, meeting in four points, the vertices, as in the figure.



We let C_i be the set of subsets of cardinality i of C_1 and we put $C=\bigcup_i C_i$. We describe the orbit structure of S_6 on C and that part of the geometry of the quadrics which is relevant for our purposes.

2.4. **Proposition.** *The orbits of the* $S_6=\Gamma/\Gamma(2)$ *action on the sets* C_n *are as follows:*

1. *The group* S_6 *acts transitively on* C_1; $\sharp C_1=10$.
2. *The group* S_6 *acts transitively on* C_2; $\sharp C_2=45$.

   Two quadrics V_m *and* V_n *intersect in a* 4*-gon of lines, thus determining a tetrahedron.*
3. *There are two orbits on* C_3, *denoted by* C_3^+ *and* C_3^-.

$$C_3=C_3^+\cup C_3^-,\qquad \sharp C_3=\binom{10}{3}=120,\qquad \sharp C_3^+=\sharp C_3^-=60.$$

A triple $\{m_1, m_2, m_3\}$ *is in* C_3^+ *if and only if* $m_1+m_2+m_3$ *is an even theta characteristic (such triples are called syzygeous).*

A triple $\{m_1, m_2, m_3\}$ *is in* C_3^- *if and only if* $m_1+m_2+m_3$ *is an odd theta characteristic (such triples are called asyzygeous).*

The quadrics of a syzygeous triple intersect in eight points, vertices of two tetrahedrons.

The quadrics of an asyzygeous triple intersect in a pair of skew lines.

4. *There are three orbits on* C_4, *denoted by* C_4^+, C_4^-, *and* C_4^*.

$$C_4 = C_4^+ \cup C_4^- \cup C_4^*, \quad \sharp C_4 = \binom{10}{4} = 210, \quad \sharp C_4^+ = \sharp C_4^- = 15, \quad \sharp C_4^* = 180.$$

A 4*-tuple* $\{m_1, m_2, m_3, m_4\}$ *is in* C_4^+ *if and only if any subtriple is in* C_3^+. (*One can also characterize* 4*-tuples in* C_4^+ *by* $m_1 + m_2 + m_3 + m_4 = 0$.) *We call such* 4*-tuples syzygeous.*

A 4*-tuple is in* C_4^- *if and only if any subtriple is in* C_3^-. *We call such* 4*-tuples asyzygeous.*

A 4*-tuple is in* C_4^* *if and only if the sum of two subtriples is even and the sum of the other two subtriples is odd.*

The sets C_4^+ *and* C_4^- *are in natural* 1-1 *correspondence with the set of the* 15 *tetrahedra and the set of* 15 *line pairs respectively as follows:*

For $S \in C_4^+$, *the complementary set* $\overline{S} \in C_6$ *consists of six characteristics whose corresponding quadrics pass through the vertices of a unique tetrahedron* T_S. *The union of the four quadrics* V_m, $m \in S$, *contains* 24 *of the* 30 *lines, but none of the six lines of the tetrahedron* T_S.

For $S \in C_4^-$, *the quadrics* V_m, $m \in S$, *all pass through a line pair* l_S. *Conversely, any of the* 15 *line pairs is cut out by four quadrics, thus setting up a* 1-1 *correspondence. The union of the four quadrics contains all the* 15 *line pairs.*

5. *There are three orbits on* C_5, *we denote them by* C_5^+, C_5^-, *and* C_5^*.

$$C_5 = C_5^+ \cup C_5^- \cup C_5^*, \quad \sharp C_5 = \binom{10}{5} = 252, \quad \sharp C_5^+ = \sharp C_5^- = 90, \quad \sharp C_5^* = 72.$$

A 5*-tuple is in* C_5^+ *if and only if it contains a* (*unique*) *syzygeous* 4*-tuple.*

A 5*-tuple is in* C_5^- *if and only if it contains a* (*unique*) *asyzygeous* 4*-tuple.*

A 5*-tuple is in* C_5^* *if and only if the sum of the five characteristics is odd.*

For any $S \in C_5^*$, *the union* $\bigcup_{m\in S} V_m$ *also contains all the* 15 *line pairs.*

6. *For* $n \geq 6$ *the orbit structure of* C_n *can be obtained by taking complements from the above. We use the notation* $C_{10-i}^+ := \{\overline{S} : S \in C_i^-\}$, $C_{10-i}^- := \{\overline{S} : S \in C_i^+\}$.

Proof. This follows easily from [8, V.6, especially Prop. 2]. The transitivity of S_6 on C_2 is in fact the corollary of Prop. 2. Note also that the sum of an even number of theta characteristics transforms linearly, so orbits may be distinguished by such a sum being 0 or not, whereas the sum of an odd number of theta characteristics transforms like a theta characteristic, so such orbits may be distinguished by the sum being even or odd. □

2.5. The complete incidence structure between points, lines, and quadrics is easily obtained, and is left to the reader as amusing passtime. We only note:

A line pair l_S, $S \in C_4^-$, lies on a quadric V_m, $m \in C_1$, if and only if $m \in S$.

Furthermore, on each line there are six points, thus on each line pair there are 12 points, and these 12 points make up three tetrahedra. Conversely, through each point there are three lines and each tetrahedron is formed out of a triple of line pairs, etc. etc....

2.6. **Lemma.** *The S_6-orbit structure on the C_i, $i = 2, \ldots, 8$, is given by*:

$$\begin{tikzcd}
 & & C_4^+(15) \arrow[r, "1"] & C_5^+(90) \arrow[r, "6"] \arrow[ddr, "2"] & C_6^+(15) \arrow[dr, "1"] & & \\
 & C_3^+(60) \arrow[ur, "4"] \arrow[dr, "2"] & & & & C_7^+(60) \arrow[dr, "4"] & \\
C_2(45) \arrow[ur, "3"] \arrow[dr, "3"] & & C_4^*(180) \arrow[r, "5"] \arrow[uur, "4"] \arrow[ddr, "4"] & C_5^*(72) \arrow[r, "2"] & C_6^*(180) \arrow[ur, "6"] \arrow[dr, "6"] & & C_8(45) \\
 & C_3^-(60) \arrow[ur, "2"] \arrow[dr, "4"] & & & & C_7^-(60) \arrow[ur, "4"] & \\
 & & C_4^-(15) \arrow[r, "1"] & C_5^-(90) \arrow[r, "6"] \arrow[uur, "2"] & C_6^-(15) \arrow[ur, "1"] & &
\end{tikzcd}$$

where $A \xrightarrow{n} B$ means: each $S \in B$ contains exactly n $S' \in A$. There is also a dual interpretation: $\overline{A} \xrightarrow{n} \overline{B}$ means: an element $S \in B$ can be extended in n ways to get an element $S' \in A$.

3. The space $X \subset P^{13}$ and its singular locus Σ

3.1. In [6], the map

$$\Theta : A_2(2, 4, 8) := \mathbb{H}_2/\Gamma_2(2, 4, 8) \longrightarrow \mathbb{P}^{13},$$
$$\tau \mapsto \left(\ldots : \theta\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}(2\tau) : \cdots : \cdots : \theta_m(\tau) : \cdots\right),$$

where m runs over the ten even charateristics and a, b run over $\{0, 1\}$, is shown to be an embedding. We denote the image by X^0 and the closure of X^0 in $\mathbb{P}^{13}$ will be denoted by X.

We define two morphisms:

$$p : X \longrightarrow \mathbb{P}^3, \qquad q : X \longrightarrow \mathbb{P}^9$$

obtained by projection on the first four and the last ten coordinates.

The map p corresponds to the natural map $A(2, 4, 8) \to A(2, 4)$; in fact, $\mathbb{P}^3$ can be identified with the Satake compactification $A^s(2, 4)$ of $A(2, 4)$. The boundary components of $A^s(2, 4)$ correspond to the 30 lines in the $\mathbb{P}^3$. The map q corresponds to the natural map $A(2, 4, 8) \to A(4, 8)$.

The equations of X are very simple. To describe them, we choose for each $m \in C_1$ a variable Z_m and consider

$$F_m := Z_m^2 - Q_m \in \mathbb{C}[\underline{X}, \underline{Z}].$$

3.2. **Lemma.** *The projective variety X has the following properties*:

1. *X is the complete intersection of the ten quadrics F_m, $m \in C_1$.*
2. *The singular locus Σ of X is exactly the inverse image of the union of the 30 lines in $\mathbb{P}^3$ under the map p. The locus Σ consists of $30 \cdot 2^3 = 240$ irreducible components, each one isomorphic to a degree-8, genus-5 curve.*

3. *X is (projectively) normal and is in fact isomorphic to the Satake compactification of $A_2(2, 4, 8)$:*

$$X \cong A_2^s(2, 4, 8).$$

Proof. Let $X' = \{(\underline{X}, \underline{Z}) \in \mathbb{P}^{13} : F_m(\underline{X}, \underline{Z}) = 0 \quad \forall m \in C_1\}$. Then by equation (2.3), we have that $X' \subset X$. Furthermore, the projection $p : X' \to \mathbb{P}^3$, $(\underline{X}, \underline{Z}) \mapsto \underline{X}$ represents X' as an iterated branched cover of $\mathbb{P}^3$, branching along the quadrics V_m, $m \in C_1$. It follows that X' is purely 10-codimensional and hence is a complete intersection.

An easy local computation shows that X' is singular exactly above the points where at least two of the quadrics V_m intersect. When we restrict to the line $X_2 = X_3 = 0$ the equations F_m reduce to

$$\text{(A)} \begin{cases} Z_1^2 = X_0^2 + X_1^2, \\ Z_2^2 = X_0^2 - X_1^2, \\ Z_5^2 = 2X_0X_1, \end{cases} \qquad \text{(B)} \begin{cases} Z_6^2 = Z_7^2 = Z_9^2 = Z_{10}^2 = 0, \\ Z_1^2 = Z_3^2, \\ Z_2^2 = Z_4^2, \\ Z_5^2 = Z_8^2. \end{cases}$$

The ideal generated by (A) defines a degree-8, genus-5 curve in $(Z_1 : Z_2 : Z_5 : X_0 : X_1)$-space (in fact, this is the elliptic-modular curve $X(8)$). The equations (B) show that the solution set consists of $2^3 = 8$ copies of this curve.

As dim Σ =1 and X' is a complete intersection, it follows that X' is irreducible and thus $X' = X$.

Furthermore, as a complete intersection is arithmetically Cohen-Macaulay, it follows from Serre's criterion for normality that X is (projectively) normal. Since the map Θ is given by modular forms, there exists a morphism $\psi: X \to A^s(2, 4, 8)$. Since $\Theta : \mathbb{H}_2/\Gamma(2, 4, 8) \to X^0$ is an isomorphism ([6, Theorem 2.2]), the map ψ is a birational isomorphism. Comparing the description of the Satake compactification in [10] with X, we see that ψ is a bijection. By Zariski's main theorem it follows that $X \cong A^s(2, 4, 8)$. □

Now let

$$I_X = (F_m : \ m \in C_1) \subset \mathbb{C}[\underline{X}, \underline{Z}], \qquad R_X := \mathbb{C}[\underline{X}, \underline{Z}]/I_X$$

(the affine coordinate ring of the cone over X). Furthermore, we let I_Σ be the ideal of (the affine cone over) the singular locus Σ with its reduced structure (i.e., I_Σ is radical).

3.3. **Lemma.** *We have* $I_\Sigma = \bigcap_{S \in C_4^-} (Z_m, \ m \in S; \ I_X)$.

Proof. Clearly, we have

$$I_\Sigma = \bigcap_{S \in C_4^-} I(l_S),$$

where $I(l_S)$ is the ideal in $\mathbb{C}[\underline{X}, \underline{Z}]$ of $p^{-1}(l_S)$, the inverse image of the line pair $l_S \subset \mathbb{P}^3$ in $\mathbb{P}^{13}$, with *reduced* structure. The ideal of a line pair l_S is

$$J(l_S) = (Q_m, \ m \in S) \subset \mathbb{C}[\underline{X}],$$

since every pair of skew lines in $\mathbb{P}^3$ is cut out by four quadrics, and the four Q_m, $m \in S$, vanish on l_S.

The ideal-theoretic inverse image of $J(l_S)$ is given by the ideal

$$\begin{aligned}\tilde{J}(l_S) &= (Q_m,\ m \in S,\ I_X) \subset \mathbb{C}[\underline{X}, \underline{Z}] \\ &= (Q_m,\ Z_m^2 - Q_m,\ m \in S,\ F_m,\ m \notin S) \\ &= (Z_m^2,\ m \in S,\ I_X).\end{aligned}$$

So, $(Z_m,\ m \in S,\ I_X) \subset \sqrt{\tilde{J}(l_S)} = I(l_S)$.

But the ideal on the left is in fact radical: by transitivity of S_6 on C_4^- we may assume $S = \{6, 7, 9, 10\}$ and then

$$\begin{aligned}&(Z_m,\ m \in S,\ I_X) \\ &\quad = (F_m,\ m \notin S,\ Z_m,\ m \in S,\ X_0X_2,\ X_1X_3,\ X_0X_3,\ X_1X_2) \\ &\quad = (F_m,\ m \notin S,\ Z_m,\ m \in S,\ X_0,\ X_1) \\ &\qquad \cap (F_m,\ m \notin S,\ Z_m,\ m \in S,\ X_2,\ X_3),\end{aligned}$$

and both of the ideals are radical (see the proof of Lemma 3.2). Thus the inclusion is actually an equality and the lemma is proved. □

4. The cusp forms

4.1. **Proposition.** *The space of cusp forms of weight three for* $\Gamma(2, 4, 8)$ *is canonically isomorphic to the degree-*6 *part of* $I_{\Sigma,X} := I_\Sigma / I_X \subset R_X = \mathbb{C}[\underline{X}, \underline{Z}]/I_X$:

$$S_3(\Gamma(2, 4, 8)) \cong I_{\Sigma, X, 6}.$$

Proof. There are in fact two 'natural isomorphisms'. We describe them both.

1. The map $\Theta: \mathbb{H}_2 \longrightarrow \mathbb{P}^{13}$ induces by pull-back an isomorphism

$$\Theta^*: H^0(X^0, \mathscr{O}_{X^0}(6)) \xrightarrow{\sim} M_3(\Gamma(2, 4, 8)).$$

As X is normal, we have $H^0(X^0, \mathscr{O}_{X^0}(6)) \cong H^0(X, \mathscr{O}_X(6))$ and because X is projectively normal

$$H^0(X, \mathscr{O}_X(6)) \cong R_{X,6}.$$

A polynomial $P \in \mathbb{C}[\underline{X}, \underline{Z}]_6$ pulls back to a cusp form if and only if it vanishes on Σ (the boundary components of X).

2. With any polynomial P of degree 6 we can associate a (meromorphic) differential form on X as follows. There is an isomorphism

$$\mathrm{Res}: \mathscr{O}_X(6) \longrightarrow \omega_X, \qquad P \mapsto \omega_P := \mathrm{Res}\left(\frac{P\Omega}{F_1 \cdots F_{10}}\right),$$

where $\Omega := \sum_{i=0}^{13} (-1)^i Y_i dY_0 \wedge \cdots \wedge \widehat{dY_i} \wedge \ldots \wedge dY_{13}$ and where the Y_i are the coordinates on $\mathbb{P}^{13}$.

The differential forms which extend holomorphically on a desingularization $\pi: \widetilde{X} \to X$ correspond, via Res, to an ideal $\mathscr{I}_A \subset \mathscr{O}_X$, which is independent of the desingularization (see [11]). We will study the 'adjunction ideal' $I_A \subset \mathbb{C}[\underline{X}, \underline{Z}]$ corresponding to $\mathscr{I}_A$.

Now a simple local calculation shows that transverse to a general point of Σ the variety X has a singularity which is isomorphic to the cone over an elliptic curve (of degree 4). This singularity 'imposes precisely one adjunction condition ($p_g = 1$)'. This means that P has to vanish on Σ if ω_P is to extend holomorphically. Therefore, $I_A \subset I_\Sigma$.

That in fact $I_A = I_\Sigma$ follows from the principle that forms which extend to the general point of the inverse image of Σ in $\widetilde{X}$ extend to all of $\widetilde{X}$ (see [4, Satz 3], [5, 'Anmerkung' to Satz III, 2.6, p. 156]). That is, the 60 special points do not impose further adjunction conditions (!). This can also be checked directly by pulling back the differential forms to an explicit resolution of singularities of X above the $2^4 \cdot 60$ special points. □

4.2. We now come to the crucial part of this paragraph: the explicit generators of the ideal $I_{\Sigma, X} \subset R_X$ or $I_\Sigma \subset \mathbb{C}[\underline{X}, \underline{Z}]$. To describe these, we need a little more notation. The tetrahedron T_S, determined by an $S \in C_4^+$, gives rise to an ideal

$$J_{T_S} \subset \mathbb{C}[\underline{X}, \underline{Z}]$$

of the functions vanishing on the six lines of T_S.

4.3. **Lemma.** *The ideal J_{T_S} is generated by four elements of degree* 3.

Proof. For each tetrahedron, the four products of three of the four linear forms defining the faces of the tetrahedron vanish on the tetrahedral lines and in fact generate the ideal. □

The following theorem allows us to find all the cusp forms of weight 3 on $\Gamma(2, 4, 8)$. We describe them in Theorem 6.4, where we also determine the Γ-action on the space $S_3(\Gamma(2, 4, 8))$.

4.4. **Theorem.** *The ideal I_Σ is (minimally) generated by the following elements:*

• F_m		$\sharp = 10$
• Z_S,	$S \in C_4^-$	$\sharp = 15$
• Z_S,	$S \in C_5^*$	$\sharp = 72$
• $Z_{S'}F$,	$F \in J_{T_S, 3}$, $S \in C_4^+$, $S' \in C_3^+$, $S' \subset S$	$\sharp = 240$.

Here we use the notation $Z_S := \prod_{m \in S} Z_m$ for any $S \in C$.

Proof. We first show that the stated elements are in the ideal I_Σ.

Because the union of the quadrics V_m, $m \in S$, contains all 30 lines in case $S \in C_4^-$ and $S \in C_5^*$, it follows that in these cases Z_S vanishes on Σ, and so $Z_S \in I_\Sigma$.

The union of the quadrics V_m, $m \in S$, $S \in C_4^+$ only contains 24 lines, which are precisely the lines not in the tetrahedron T_S determined by S. Furthermore, the union of any three of the four quadrics V_m, $m \in S$, $S \in C_4^+$, already contains the same 24 lines. Consequently, multiplying any $Z_{S'}$, $S' \in C_3^+$, $S' \subset S$ with any element of J_{T_S} will give a function vanishing on the whole of Σ.

The difficult part of the theorem is to show that there is nothing more in I_Σ. So far, this depends on an explicit computation of the intersection of the 15 ideals $I(l_S)$, $S \in C_4^-$. For this we used the computer program 'Macaulay'. The computer output consisted of 337 elements, generating this ideal, which were readily recognized as the elements above.

To give a computer independent proof, it seems necessary to understand the combinatorics much better. □

4.5. **Corollary**. *We have*

1. $\dim(I_{\Sigma,X,4}) = 15$,
2. $\dim(I_{\Sigma,X,5}) = 282$,
3. $\dim(I_{\Sigma,X,6}) = 2283$.

Proof. It is convenient to use the following isomorphism:

$$R_X = \bigoplus_{S\in C} \mathbb{C}[\underline{X}]Z_S,$$

stating that, modulo the F_m, every polynomial can be reduced in a *unique* way to a sum of squarefree monomials Z_S, with coefficients in $\mathbb{C}[\underline{X}]$. (A Cohen-Macaulay ring is a free module over a parameter system.) □

From Theorem 4.4 we have

$$I_{\Sigma,X,4} = \bigoplus_{S\in C_4^-} \mathbb{C}Z_S, \qquad \text{so} \quad \dim(I_{\Sigma,X,4}) = 15.$$

In degree 5 we thus find, apart from the 72 new generators Z_S, $S \in C_5^*$, the elements of $I_{\Sigma,X,4}$ multiplied by a linear factor. The following cases occur:

1. Z_mZ_S, $\quad m \notin S$,
2. Z_mZ_S, $\quad m \in S$,
3. X_iZ_S, $\quad i = 0, 1, 2, 3$.

From diagram 2.6 we see that

in case (1)	$Z_mZ_S = Z_{S'}$,	$S' \in C_5^-$	$\sharp = 90$
in case (2)	$Z_mZ_S = Q_mZ_{S'}$,	$S' \in C_3^-$, $\{m, S'\} \in C_4^-$	$\sharp = 60$
in case (3)	X_iZ_S		$\sharp = 4 \cdot 15 = 60.$

Altogether, in degree 5 we find $72 + 90 + 60 + 60 = 282$ monomials.

To get elements of degree 6, we proceed in the same way: apart from the 240 generators $FZ_{S'}$, $F \in J_{T_S}$, $S' \subset S \in C_4^+$, we get all the other factors by multiplying something of degree 5 with a linear factor. Starting from the 90 elements Z_S, $S \in C_5^-$, we get:

1. Z_mZ_S, $m \notin S$,
2. Z_mZ_S, $m \in S$,
3. X_iZ_S, $i = 0, 1, 2, 3$.

In case (1) we have $Z_mZ_S = Z_{S'}$, $S' \in C_6^-$, $\sharp = 15$. In case (2) there are two subcases: (2a) $S' := S - \{m\} \in C_4^-$, $Z_mZ_S = Q_mZ_{S'}$, $\sharp = 360$ and (2b) $S' \in C_4^*$, $\sharp = 360$. In case (3) we find $4 \cdot 90 = 360$ elements.

Proceeding in this way with the other elements of degree 5 in the ideal, we get the following table (the last column relates them to the representation studied in §6):

elements		dim	representation
Z_S	$S \in C_6^-$	15	R_6^-
$Q_m Z_S$	$S \in C_4^-$, $\{m, S\} \in C_5^-$	90	$R_4^-(0; 2)$
$Q_m Z_S$	$S \in C_4^*$, $\{m, S\} \in C_5^-$	360	$R_4^-(1; 1)$
$X_i Z_S$	$S \in C_5^-$	360	$R_4^-(0; 1)$
$Q_m Q_{m'} Z_S$	$S \in C_2$, $\{m, m', S\} \in C_4^-$	90	$R_4^-(1, 1; 0)$
$X_i Q_m Z_S$	$S \in C_3^-$, $\{m, S\} \in C_4^-$	240	$R_4^-(1; 0)$
$X_i X_j Z_S$	$S \in C_4^-$	150	$R_4^-(0; 2) \oplus R_4^-(2; 0)$
Z_S	$S \in C_6^*$	180	R_6^*
$Q_m Z_S$	$S \in C_4^*$, $\{m, S\} \in C_5^*$	360	$R_5^*(1; 0)$
$X_i Z_S$	$S \in C_5^*$	288	R_5^*
$F Z_S$	$F \in J_{T_{S'}, 3}$, $S \subset S' \in C_4^+$	240	$R(3, 3)$

In particular, we find $\dim I_{\Sigma, X, 6} = \dim S_3(\Gamma(2, 4, 8)) = 2283$.

5. The theta transformation formula

5.1. Let Γ' be a normal subgroup of Γ and let $M_3(\Gamma')$ be the space of modular forms of weight 3 on Γ'. The group Γ (in fact Γ/Γ') acts on $M_3(\Gamma')$ by

$$f \mapsto f|M, \qquad \text{with} \quad (f|M)(\tau) = \det(C\tau + D)^{-3} f(M \cdot \tau).$$

To decompose the spaces of cusp forms with respect to this representation, we introduce the following symplectic matrices:

$$e_1(n) = \begin{pmatrix} 1 & & & \\ 2n & 1 & & \\ & & 1 & -2n \\ & & & 1 \end{pmatrix}, \qquad e_3(n) = \begin{pmatrix} 1 & & & 2n \\ & 1 & 2n & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$e_2(n) = {}^t e_1, \qquad e_4(n) = {}^t e_3,$$

$$e_5(n) = \begin{pmatrix} A & \\ & {}^t A^{-1} \end{pmatrix}, \qquad e_6(n) = \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix},$$

$$e_7(n) = \begin{pmatrix} 1 & & 2n & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad e_8(n) = \begin{pmatrix} 1 & & & \\ & 1 & & 2n \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$e_9(n) = {}^t e_7(n), \qquad e_{10}(n) = {}^t e_8(n).$$

Here, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is some matrix in $\mathrm{SL}_2(\mathbb{Z})$ which is congruent to

$$\begin{pmatrix} 1 + 2n & 0 \\ 0 & 1 + 2n \end{pmatrix}$$

modulo $4n$.

To find the action of Γ on the modular forms, we use the transformation formula for theta functions ([8, V, §1, Corollary, p. 176 and V, §2, Theorem 3, p. 182]):

5.2. **Lemma** (Igusa's transformation formula). *For* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_{2g}(\mathbb{Z})$ *and* $m \in \mathbb{R}^{2g}$ *a theta characteristic, we define*

$$M \cdot m := mM^{-1} + \tfrac{1}{2}((C^tD)_0\,(A^tB)_0),$$

with $(C^tD)_0$ *the diagonal of* C^tD. *Define*

$$\phi_m(M) = -\tfrac{1}{2}\left(m'^tDB^tm' - 2m'^tBC^tm'' + m''^tCA^tm'' - (m'^tD - m''^tC)^t(A^tB)_0\right).$$

Then

$$\theta_{M\cdot m}((A\tau+B)(C\tau+D)^{-1}) = \kappa(M)\exp(2\pi i\phi_m(M))\sqrt{\det(C\tau+D)}\,\theta_m(\tau),$$

in which $\kappa(M)$ *is a complex number of absolute value* 1 *which depends only on* M *and the choice of the square root. In particular, it does not depend on the characteristic* m. *Thus,* $\kappa(M)^2$ *is well defined; for* $M \in \Gamma_g(2)$ *one has*

$$\kappa(M)^2 = (-1)^{\operatorname{trace}(D-1)/2}.$$

5.3. In the remainder of this paragraph we derive two lemmas from this formula. The first lemma gives an explicit form of the transformation formula for certain matrices. The second lemma studies the transformation behavior of the functions $\theta\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right](2\tau)$ which are an ingredient of some of the cusp forms.

5.4. **Lemma.** *For every half-integral characteristic*

$$m = \tfrac{1}{2}(a, b, c, d) \qquad \textit{with} \quad a, b, c, d \in \{0, 1\}$$

and every M $(\in \Gamma(2))$ *as below we have*

$$\theta_m(M\cdot\tau) = \chi_m(M)\theta_m(\tau)$$

for all $\tau \in \mathbb{H}$ *and with* $\chi_m(M)$ *as in the table.*

M	$e_1(1)$	$e_2(1)$	$e_3(1)$	$e_5(1)$	$e_6(1)$	$e_7(1)$	$e_8(1)$
$\chi_m(M)$	$(-1)^{bc}$	$(-1)^{ad}$	$(-1)^{ab}$	1	$(-1)^{ac}$	i^a	i^b

Proof. We will write $m_1 := (a, b)$, $m_2 := (c, d)$. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(2)$ with $A = D = I$, $B = 2B'$, $C = 0$; note that B' is then a symmetric matrix with integral coefficients. We find

$$\begin{aligned}
\theta_m(M\cdot\tau) &= \theta_m(\tau + B)\\
&= \sum_k \exp(2\pi i[\tfrac{1}{2}(k + \tfrac{m_1}{2})(\tau + B)^t(k + \tfrac{m_1}{2}) + (k + \tfrac{m_1}{2})^t(\tfrac{m_2}{2})])\\
&= \sum_k \exp(2\pi i[\tfrac{1}{2}(k + \tfrac{m_1}{2})\tau^t(k + \tfrac{m_1}{2}) + kB''k + kB'm_1 + \tfrac{m_1}{2}B'\tfrac{m_1}{2}\\
&\qquad\qquad + (k + \tfrac{m_1}{2})^t(\tfrac{m_2}{2})])\\
&= \exp(\tfrac{2\pi i}{4}m_1B'm_1)\cdot\theta_m(\tau).
\end{aligned}$$

From this, $\chi_m(M)$ for $M = e_3, e_7, e_8$ is easily computed.

Let now $M = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$; this implies that $D = {}^tA^{-1}$. Then

$$\begin{aligned}
\theta_m(M\cdot\tau) &= \theta_m(A\tau{}^tA) \\
&= \sum_k \exp(2\pi i[\tfrac{1}{2}(k+\tfrac{m_1}{2})A\tau^t A^t(k+\tfrac{m_1}{2}) + (k+\tfrac{m_1}{2})AA^{-1}(\tfrac{m_2}{2})]) \\
&= \sum_k \exp(2\pi i[\tfrac{1}{2}((k+\tfrac{m_1}{2})A)\tau^t((k+\tfrac{m_1}{2})A) + ((k+\tfrac{m_1}{2})A)^t((\tfrac{m_2}{2})^t A^{-1})]) \\
&= \theta_n(\tau),
\end{aligned}$$

the characteristic n being given by

$$n = (n_1, n_2), \qquad n_1 = m_1 A, \qquad n_2 = n_2{}^tA^{-1}.$$

In case $A = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, so $M = e_1(1)$, one obtains:

$$n = (a, b, c, d) + (2b, 0, 0, -2c), \qquad \text{thus} \quad \theta_n(\tau) = (-1)^{bc}\theta_m(\tau),$$

where we use (θ.2) from [8, p. 49]. The formula for $\chi_m(M)$, with $M = e_2(1)$, $e_5(1)$, $e_6(1)$, is derived analogously; note one may take $A = -I$ in e_5 and e_6.

Note that by comparing this result with Lemma 5.2, we find

$$\kappa(M)\sqrt{\det(C\tau + D)} = 1$$

for these matrices. □

5.5. **Lemma.** *Let $M \in \Gamma_g(2)$ with*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I + 2A' & 2B' \\ 2C' & I + 2D' \end{pmatrix};$$

let $m = (m_1, \dots, m_g)$ with $m_i \in \{0, 1\}$ and let $\tau \in \mathbb{H}_g$.

Then we have

$$\theta[{}^m_0](2M\cdot\tau) = \lambda(M, \tau)\cdot(-1)^{(m+y)\cdot{}^tx}\theta[{}^{m+y}_{\ 0}](2\tau),$$

with $\lambda(M, \tau)$ ***independent*** *of the characteristic m and*

$$\lambda(M, \tau)^2 = \det(C\tau + D), \quad x := \operatorname{diag}(B'), \quad y := \operatorname{diag}(C'),$$

where we view the diagonals as row vectors.

In case $C = 0$, we have $\lambda(M, \tau) = 1$.

Proof. This is actually a special case of Igusa's transformation formula 5.2. Indeed,

$$\begin{aligned}
\theta[{}^m_0](2(M\cdot\tau)) &:= \theta[{}^m_0](2(A\tau + B)(C\tau + D)^{-1}) \\
&= \theta[{}^m_0]((A(2\tau) + 2B)(C'(2\tau) + D)^{-1}) \\
&= \theta[{}^m_0](M'\cdot(2\tau)).
\end{aligned}$$

One easily verifies that the matrix M' is also a symplectic matrix with integral coefficients, thus M'^{-1} is easy to compute:

$$M' := \begin{pmatrix} A & 2B \\ C' & D \end{pmatrix}, \qquad M'^{-1} := \begin{pmatrix} {}^tD & -2{}^tB \\ -{}^tC' & {}^tA \end{pmatrix}.$$

The action of M' on the characteristic $(\frac{m}{2}, 0)$ is then given by

$$\begin{aligned} M' \cdot (\tfrac{m}{2}, 0) &:= (\tfrac{m}{2}, 0)M'^{-1} + \tfrac{1}{2}((C'{}^tD)_0, \ 2(A^tB)_0) \\ &= (\tfrac{m}{2}, 0)\begin{pmatrix} {}^tD & -2\,{}^tB \\ -{}^tC' & {}^tA \end{pmatrix} + \tfrac{1}{2}((C'{}^tD)_0, \ 2(A^tB)_0) \\ &= (\tfrac{m}{2}{}^tD, \ -m^tB) + \tfrac{1}{2}((C'{}^tD)_0, \ 2(A^tB)_0) \\ &= (\tfrac{m+y}{2}, \ 0) + (m^tD' + y', \ 2(-m^tB' + (A^tB')_0) \\ &= k + l. \end{aligned}$$

Using (θ.2) from [8, p. 49], we find $\theta_{k+l} = \theta_k$. It is then easy to verify that

$$\theta[{}^{m}_{0}](2M \cdot \tau) = \theta_{M'.k}(M' \cdot (2\tau)).$$

Applying 5.2 to the right-hand side, we find

$$\theta[{}^{m}_{0}](2M \cdot \tau) = \lambda(M, \tau)\exp(2\pi i \phi_k(M'))\theta_k(2\tau).$$

Since $k = (\frac{m+y}{2}, 0)$, the only nonzero terms in $\phi_k(M')$ are

$$\begin{aligned} \phi_k(M') &= -\tfrac{1}{4}(m+y)({}^tDB)^t(m+y) + \tfrac{1}{2}(m+y)^tD(A^tB)_0 \\ &\in \tfrac{1}{2}(m+y)^tx + \mathbb{Z}, \end{aligned}$$

where we use that $B = 2B'$, $D = I + 2D'$ and $x := \operatorname{diag}(B')$.

In case $C = 0$ it follows from (the proof of) Lemma 5.4 that $\lambda(M, \tau) = 1$. This completes the proof of Lemma 5.5. □

6. The representation of $\Gamma/\Gamma(2, 4, 8)$ on $S_3(\Gamma(2, 4, 8))$

6.1. Recall that the subgroup $\Gamma(2)$ fixes the characteristics. For $f = \prod_{i=1}^{2k} \theta_{m_i}$, a modular form of weight k on $\Gamma(4, 8)$, we can then define the homomorphism:

$$\chi_f : \Gamma(2)/\Gamma(2, 4, 8) \longrightarrow \mathbb{C}^*, \qquad \text{by} \qquad f|M = \chi_f(M)f.$$

As Lemma 5.5 shows, $\Gamma(2)$ does not fix the $\theta[{}^{a}_{0}{}^{b}_{0}](2\tau)$'s. We define a subgroup $\Gamma'(2)$ by

$$\Gamma'(2) := \left\{ \begin{pmatrix} A & B \\ 2C' & D \end{pmatrix} \in \Gamma(2) : \ \operatorname{diag}(C') \equiv 0 \bmod 2 \right\}.$$

For $g = \theta[{}^{e_1}_{0}{}^{e_2}_{0}](2\tau)\prod_{i=1}^{2k-1} \theta_{m_i}(\tau)$, a modular form of weight k on $\Gamma(2, 4, 8)$, we define a character

$$\chi_g : \Gamma'(2)/\Gamma(2, 4, 8) \longrightarrow \mathbb{C}^*, \qquad \text{by} \qquad g|M = \chi_g(M)g.$$

The following proposition lists some character values.

6.2. **Proposition.** *Let m_i be a half-integral characteristic with $m_i := \frac{1}{2}(a_i, b_i, c_i, d_i)$.*

(a) *Let*

$$f(\tau) = \theta_{m_1}(\tau)\theta_{m_2}(\tau)\cdots\theta_{m_{2k}}(\tau).$$

Then f is a modular form of weight k on $\Gamma(4, 8)$ and values of χ_f are listed below. One also has $\chi_f(e_5(1)) = 1$.

M	$e_1(1)$	$e_2(1)$	$e_3(1)$	$e_4(1)$	$e_6(1)$	$e_7(1)$	$e_8(1)$	$e_9(1)$	$e_{10}(1)$
$\chi_f(M)$	$(-1)^{\Sigma b_j c_j}$	$(-1)^{\Sigma a_j d_j}$	$(-1)^{\Sigma a_j b_j}$	$(-1)^{\Sigma c_j d_j}$	$(-1)^{1+\Sigma a_j c_j}$	$i^{\Sigma a_j}$	$i^{\Sigma b_j}$	$i^{\Sigma c_j}$	$i^{\Sigma d_j}$

(b) *Let* $e_1, e_2 \in \{0, 1\}$ *and let*

$$g(\tau) = \theta[\begin{smallmatrix} e_1 & e_2 \\ 0 & 0 \end{smallmatrix}](2\tau)\theta_{m_1}(\tau)\theta_{m_2}(\tau)\cdots\theta_{m_{2k-1}}(\tau).$$

Then g *is a modular form of weight* k *on* $\Gamma(2, 4, 8)$, *and some values of* χ_g *are listed below. One also has* $\chi_g(e_5(1)) = 1$.

$e_1(1)$	$e_2(1)$	$e_3(1)$	$e_4(1)$	$e_6(1)$	$e_7(1)$	$e_8(1)$	$e_9(2)$	$e_{10}(2)$
$(-1)^{\Sigma b_j c_j}$	$(-1)^{\Sigma a_j d_j}$	$(-1)^{\Sigma a_j b_j}$	$(-1)^{\Sigma c_j d_j}$	$(-1)^{1+\Sigma a_j c_j}$	$i^{2e_1+\Sigma a_j}$	$i^{2e_2+\Sigma b_j}$	$(-1)^{\Sigma c_j}$	$(-1)^{\Sigma d_j}$

Proof. That $f \in M_3(\Gamma(4, 8))$ follows from the corollary in [8, V.7]. Note that for the matrix A in e_5 and e_6 we can take $A = -I$. Then $e_5(1) = -I$, which acts trivially on $\mathbb{H}_2$. The lemma then follows easily from 5.2. Note that all matrices M except $e_6(1)$ have trace$(D - I) = 0$, so $\kappa(M)^2 = 1$ by [8, V.3, Theorem 3], and that $\kappa(e_6(1))^2 = -1$.

For the second part, we oberve that Lemmas 5.5 and 5.2 imply that $\Gamma(2, 4)$ acts by a character on the modular form g. In [6] it is shown that this character is trivial on $\Gamma(2, 4, 8)$ (but it is not trivial on $\Gamma(4, 8)$). Therefore, g is a modular form on $\Gamma(2, 4, 8)$.

The matrices $e_1, e_2, e_3, e_5, e_7, e_8$ have '$C = 0$', so χ_g can be computed directly from 5.2 and Lemmas 5.4 and 5.5. For the matrix A in e_5 and e_6 we can take $A = -I$. Therefore, also e_6 has '$C = 0$'.

The remaining matrices e_4, e_9, e_{10} are of the form $M = (\begin{smallmatrix} I & 0 \\ 2C' & I \end{smallmatrix})$, so $D = I$, and $\kappa(M)^2 = 1$. In the formula for $f|M$ there appears however the constant $\kappa(M')\kappa(M)^{2k-1}$, with $M' = (\begin{smallmatrix} I & 0 \\ C' & I \end{smallmatrix})$, cf. the proof of Lemma 5.5.

To find $\kappa(M')\kappa(M)$, we compute $\theta_0(2M\tau)\theta_0(M\tau)$. Note that

$$M = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & -C \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

First of all, we find

$$\theta_0(-2\tau^{-1})\theta_0(-\tau^{-1}) = \tfrac{1}{2}\det(\tau)\theta_0(\tau/2)\theta_0(\tau).$$

Up to a 4th root of unity this follows directly from 5.2. By specializing $\tau = (\tau_{kl})$ to a matrix with $\tau_{kl} = 0$ if $k \neq l$, we get $\theta[\begin{smallmatrix} e_1 & e_2 \\ f_1 & f_2 \end{smallmatrix}](\tau) = \theta[\begin{smallmatrix} e_1 \\ f_1 \end{smallmatrix}](\tau_{11})\theta[\begin{smallmatrix} e_2 \\ f_2 \end{smallmatrix}](\tau_{22})$. The formula then follows from the identity

$$\theta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](-\tau_1^{-1}) = \sqrt{-i\tau_1}\cdot\theta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](\tau_1), \qquad \text{with } \operatorname{Re}(\sqrt{-i\tau_1}) > 0, \quad \tau_1 \in \mathbb{H}_1.$$

Next we apply $(\begin{smallmatrix} I & -C \\ 0 & I \end{smallmatrix}) = (\begin{smallmatrix} I & -2C' \\ 0 & I \end{smallmatrix})$:

$$\tfrac{1}{2}\det(\tau - C)\theta_0(\tfrac{\tau}{2} - C')\theta_0(\tau - 2C') = \tfrac{1}{2}\det(\tau - C)\theta_0(\tfrac{\tau}{2})\theta_0(\tau)\ ,$$

where we use Lemma 5.5. Applying $(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix})$, we obtain

$$\tfrac{1}{2}\det(-\tau^{-1} - C)\theta_0(-(2\tau)^{-1})\theta_0(-\tau^{-1}) \;=\; \det(C\tau + I)\theta_0(2\tau)\theta_0(\tau).$$

Comparison with 5.2 shows that $\kappa(M')\kappa(M) = 1$. The values of $\chi_g(M)$ then follow from 5.2. (Note that in [6] it is proved that $\kappa(M')\kappa(M) = -1$ with $M = e_6(2)$, the generator of $\Gamma(4, 8)/\Gamma(2, 4, 8)$, thus some care is needed.) □

6.3. We will now determine the splitting of $S_3(\Gamma(2,4,8))$ into irreducible Γ-representations and we show that the characters χ_f, χ_g determine the cusp forms (within the space of cusp forms). This will be important when we study the Hecke action in the next section.

6.4. **Theorem.** *The space $S_3(\Gamma(2,4,8))$ is the direct sum of 11 irreducible Γ-representations. The repesentation on $S_3(\Gamma(4))$ is irreducible, and $S_3(\Gamma(4,8))$ is the direct sum of $7 = 1 + 6$ irreducible representations.*

Below we label the representations, their dimensions and a cusp form in each representation space.

space	label	dim	cusp form
$S_3(\Gamma(4))$	R_6^-	15	$f_1 = \theta[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}1&1\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&1\\1&0\end{smallmatrix}][\begin{smallmatrix}1&1\\1&1\end{smallmatrix}]$
	$R_4^-(0;2)$	90	$f_2 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}]$
	$R_4^-(1,1;0)$	90	$f_3 = \theta[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}]$
	$R_4^-(1;1)$	360	$f_4 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}1&0\\0&1\end{smallmatrix}]$
	R_6^*	180	$f_5 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\1&1\end{smallmatrix}][\begin{smallmatrix}0&1\\1&0\end{smallmatrix}][\begin{smallmatrix}1&1\\1&1\end{smallmatrix}]$
	$R_4^-(2;0)$	60	$f_6 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&1\end{smallmatrix}][\begin{smallmatrix}0&1\\1&0\end{smallmatrix}]$
$S_3(\Gamma(4,8))$	$R_5^*(1;0)$	360	$f_7 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\1&1\end{smallmatrix}]$
	$R_4^-(1;0)$	240	$g_1 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}](2\tau)\theta[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}](\tau)$
	$R_4^-(0;1)$	360	$g_2 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}](2\tau)\theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}](\tau)$
	R_5^*	288	$g_3 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}](2\tau)\theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\tau)$
$S_3(\Gamma(2,4,8))$	$R(3,3)$	240	$g_4 = \theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}][\begin{smallmatrix}1&0\\0&0\end{smallmatrix}][\begin{smallmatrix}0&1\\0&0\end{smallmatrix}](2\tau)\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}][\begin{smallmatrix}0&0\\0&1\end{smallmatrix}][\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\tau)$

So the first representation is equal to $S_3(\Gamma(4))$ and the sum of the first seven representations is $S_3(\Gamma(4,8))$.

The space $S_3(\Gamma(4,8))$ is a direct sum of one-dimensional spaces $\mathbb{C}f$, where f is a monomial, i.e., a product of six theta constants, and for monomials f, $f' \in S_3(\Gamma(4,8))$ we have

$$\chi_f = \chi_{f'} \Longleftrightarrow f = f'.$$

The space $S_3(\Gamma(2,4,8)) = S_3(\Gamma(4,8)) \oplus W'$, where W' is spanned by linear combinations of monomials, which are products of one $\theta[\begin{smallmatrix}a&b\\0&0\end{smallmatrix}](2\tau)$ and five $\theta_m(\tau)$'s. Under the action of $\Gamma'(2)$, the space W' is a direct sum of mutually distinct one-dimensional subrepresentations:

$$W' = \bigoplus_g \mathbb{C}g, \quad \textit{and} \quad \chi_g = \chi_{g'} \Longleftrightarrow g = g'.$$

Proof. The meaning of the names of the representation spaces is as follows: $R_4^-(1,1;0)$ is the space obtained by taking (linear combinations of) all products of one of the 15 monomials θ_S, $S \in C_4^-$, and squaring two different terms occurring. Thus we get $15 \cdot \binom{4}{2} = 90$ different monomials. Note that Theorem 4.4 implies that θ_S is a cusp form. Similarly, $R_4^-(1;1)$ is spanned by multiplying a monomial θ_S, $S \in C_4^-$, by a θ_m with $m \in S$ and a θ_n with $n \notin S$. The dimension of this space is then $15 \cdot 4 \cdot 6 = 360$. The meaning of the other terms is similar.

It follows from Theorem 4.4 that the 11 spaces are contained in $S_3(\Gamma(2, 4, 8))$ and that $S_3(\Gamma(2, 4, 8))$ is in fact a direct sum of these spaces.

Using Proposition 2.4 (and also [8, V.6] if the six characteristics are not distinct), it is not hard to verify that the first seven spaces are stable under the Γ-action and that Γ permutes the monomials in each space transitively. Since only the monomials without a $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}]$ are on $\Gamma(4, 8)$, we see that $S_3(\Gamma(4, 8))$ is spanned by the monomials in the orbits of $f_1, \ldots, f_7$. In particular, dim $S_3(\Gamma(4, 8)) = 1155$.

To show the irreducibility of the seven subrepresentations in $S_3(\Gamma(4, 8))$, we actually need that $\chi_f\colon \Gamma(2) \to \mathbb{C}^*$ determines f in $S_3(\Gamma(4, 8))$. To prove it, we use the Γ-action, so we may assume that $f = f_i$, with f_i one of the seven forms listed. Then one determines all 6-tuples of characteristics which give the same character and one observes, for each i, that there is only one set of characteristics which gives a cusp form (to wit, the set of characteristics of f_i itself).

Suppose now that a linear combination of monomials from $S_3(4, 8)$ lies in a subrepresentation. Then, using the action of $\Gamma(2)$ and taking linear combinations, we see that each monomial in the combination lies in that subrepresentation. As each monomial in $S_3(4, 8)$ lies in the Γ-span of one of the seven cusp forms listed, the subrepresentation is a direct sum of some of the seven listed. Therefore, $S_3(\Gamma(4, 8))$ is a direct sum of seven irreducible Γ-representations.

Since f_1 is invariant under $\Gamma(4)$, we have $f_1 \in S_3(\Gamma(4))$. Since the orbit defining the 6-tuple of f_1 is C_6^-, which has 15 elements, and dim $S_3(\Gamma(4)) = 15$ [12], it follows that $S_3(\Gamma(4))$ is spanned by f_1.

We now consider all of $S_3(\Gamma(2, 4, 8))$. Let W be the subspace of $M_3(\Gamma(2, 4, 8))$ spanned by products of one $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ and five $\theta_m(\tau)$'s. Using $\theta_m^2(\tau) = Q_m(\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau))$, we also have that the cusp form g_4 is in W, in fact

$$\theta[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}][\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}](2\tau) = \tfrac{1}{4}(\theta^2[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}](\tau) - \theta^2[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}](\tau)).$$

Similarly, all of $R(3, 3)$ is contained in W. Therefore,

$$S_3(\Gamma(2, 4, 8)) = S_3(\Gamma(4, 8)) \oplus W', \qquad \text{with} \quad W' := W \cap S_3(\Gamma(2, 4, 8)).$$

(Indeed, a product of one $\theta_m(2\tau)$ and five $\theta_n(\tau)$'s is never in $S_3(\Gamma(4, 8))$.)

Under the action of Γ, the four $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ are mapped to linear combinations of these four theta nulls (cf. [8, II.5, Theorem 6]), whereas the $\theta_m(\tau)$'s are permuted. The space W is thus stable under the action of Γ. Since the 5-tuples of characteristics in the cusp forms $g_1, \ldots, g_4$ are in different orbits for the Γ-action, we already find four distinct subrepresentations in $W' = S_3(\Gamma(2, 4, 8)) \cap W$, each spanned by the Γ-transforms of a g_i.

To see that these four subrepresentations span W', let $f \in W'$ be the product of one $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ and five θ_m's. Then there is a transformation in Γ which maps the five θ_m's to the five θ_m's of one of the first three cusp forms. Therefore, in the subrepresentation generated by f there is a linear combination of the $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$'s multiplied by the product of the five θ_m from such a cusp form. Using the action of $e_7(1)$ and $e_8(1)$, we find that, for some a, b, $\theta[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}](2\tau)$ times the product of the same five θ_m lies in the subrepresentation generated by f (cf. Lemma 5.5). Applying $e_7(1)^a e_{10}(1)^b$, which is in $\Gamma(2)$

and thus fixes the five characteristics but acts on the other (see Lemma 5.5), we get $\theta[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}](2\tau)$ times the same product of the five θ_m, i.e., one of the g_i ($i = 1, 2, 3$). The monomials from $R(3, 3)$ are in fact permuted transitively (up to a scalar multiple), as can be seen from the geometry of the tetrahedron or by a similar argument of above.

To prove the irreducibility of these four representations, we need that χ_g determines the cusp form $g \in W'$. This is done as in the $S_3(\Gamma(4, 8))$ case by explicit verification. In fact, for a monomial g obtained from a g_i, $i = 1, 2, 3$, the restriction of χ_g to $\Gamma(2, 4)$ determines the five θ_m among the possible 5-tuples obtained in this way. Since Γ has three orbits (coresponding to the i) on these 5-tuples and $\Gamma(2, 4)$ is a normal subgroup, one need only verify that the 5-tuples of the g_i are uniquely determined by their character. The action of e_7 and e_8 allows one to recover the $\theta[\begin{smallmatrix}a&b\\0&0\end{smallmatrix}](2\tau)$ from χ_g. Similarly, using the action of Γ on the Γ-orbit of g_4, one need only check that g_4 is determined by its character.

The irreducibility of the four representations is then proven as in the $S_3(\Gamma(4, 8))$ case. □

7. Hecke eigenforms

7.1. The Hecke algebra, generated by the Hecke operators T_p and T_{p^2} for primes $p > 2$, acts on the space $S_3(\Gamma(2, 4, 8))$. We want to determine a basis of eigenvectors. For an eigenform f and a prime $p > 2$ such that

$$T_p f = \lambda_p f, \qquad T_{p^2} f = \lambda_{p^2} f$$

one defines the Hecke polynomial

$$H_p(X) := X^4 - a_p X^3 + a_{p^2} X^2 - a_p p^3 X + p^6, \qquad \text{with} \quad \begin{cases} a_p = \lambda_p, \\ a_{p^2} = \lambda_p^2 - \lambda_{p^2} - p^2. \end{cases}$$

7.2. For modular forms on $\Gamma(8)$ there appears a character $\chi_2 : (\mathbb{Z}/8\mathbb{Z})^* \to \{\pm 1\}$ in the Hecke polynomial. This character is defined by $f|M_p = \chi_2(p) f$ with $M_p \in \Gamma$ a matrix with $M_p \equiv \operatorname{diag}(p^{-1}, p^{-1}, p, p) \bmod 8$. We will show that χ_2 is trivial for modular forms on $\Gamma(2, 4, 8)$.

If $p \equiv -1 \bmod 8$, then one may take $M_p = -I$, and thus $\chi_2(p) = +1$ since $-I$ acts trivially on f. If $p \equiv 5 \bmod 8$, then put $A = (\begin{smallmatrix}5&8\\8&13\end{smallmatrix}) \in SL_2(\mathbb{Z})$ and take $M_p := (\begin{smallmatrix}A&0\\0&{}^tA^{-1}\end{smallmatrix})$. Since $M_p \in \Gamma(2, 4, 8)$, it also acts trivially on f. Therefore the character χ_2 is trivial (for any modular form on $\Gamma(2, 4, 8)$).

7.3. The action of the Hecke operators is given by the formulas in [2]. In fact, if

$$f(\tau) = \sum_N a_N \exp\left(\tfrac{2\pi i}{8} \operatorname{trace}(N\tau)\right), \quad (T_{p^i} f)(\tau) = \sum_N b_N \exp\left(\tfrac{2\pi i}{8} \operatorname{trace}(N\tau)\right),$$

are the Fourier-Jacobi series of f and $T_{p^i} f$, then explicit formulas expressing b_N in terms of a_N and p^i are given in [2]. We will write

$$a_N = a(n, r, m), \qquad \text{with} \quad N = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$$

a positive definite half-integral matrix.

In case the quadratic form $nx^2+rxy+my^2$ has no nontrivial zeros mod p, then one simply has $b(n,r,m)=a(p^i n, p^i r, p^i m)$. We used this fact often in our computations.

To find the eigenspaces for the Hecke action, we will use the following proposition. It is an easy generalization of Satz 32 from [7].

7.4. **Proposition.** *Let Γ be a subgroup of $Sp_{2g}(\mathbb{Z})$, with $\Gamma_g(q)\subset\Gamma\subset Sp_{2g}(\mathbb{Z})$. Define*

$$\overline{\Gamma}:=\pi(\Gamma), \qquad \textit{with} \quad \pi\colon \mathrm{Sp}_{2g}(\mathbb{Z})\longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/q\mathbb{Z})$$

the reduction map. Assume that for every $n\in(\mathbb{Z}/q\mathbb{Z})^*$ and every $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\overline{\Gamma}$ one has that $\left(\begin{smallmatrix}A&n^{-1}B\\nC&D\end{smallmatrix}\right)\in\overline{\Gamma}$.

Then the Hecke operator T_n maps the space

$$M_k(\Gamma,\chi):=\{f\in M_k(\Gamma_g(q))\colon f|M=\chi(M)f, \quad \forall M\in\Gamma\}$$

to the space $M_k(\Gamma,\chi')$, where $\chi'\colon\overline{\Gamma}\to C^*$ is given by

$$\chi'(M):=\chi(M'), \qquad \textit{with} \qquad M'\in\Gamma, \quad M'\equiv\begin{pmatrix}A&n^{-1}B\\nC&D\end{pmatrix} \mod q.$$

Proof. The Hecke operator T_n is defined as a sum:

$$T_nf:=\sum_k f|H_k, \qquad H_k\equiv D_n:=\left(\begin{smallmatrix}I&0\\0&nI\end{smallmatrix}\right) \bmod q$$

and where $\Gamma(q)D_n\Gamma(q)=\coprod_{k\in J}\Gamma(q)H_k$, a disjoint union. By [3, Lemma 1.1 (2)], one then also has

$$\Gamma D_n\Gamma=\coprod_{k\in J}\Gamma H_k.$$

For any $M\in\Gamma$, the matrices H_kM, $k\in J$, are then also a set of coset representatives. Therefore, there is a permutation $\sigma=\sigma_M\colon J\to J$ and there are $M_k\in\Gamma$ such that

$$H_kM=M_kH_{\sigma(k)}, \qquad \text{and thus} \quad M_k\equiv D_nMD_n^{-1} \bmod q.$$

Given M, the matrices M_k are thus all congruent mod q to a matrix M'. By the assumption on Γ, we can choose $M'\in\Gamma$. Therefore, for $f\in M_k(\Gamma,\chi)$ we obtain

$$\begin{aligned}(T_nf)|M&=\sum_k f|(H_kM)=\sum_k f|M_kH_{\sigma(k)}\\&=\sum_k\chi(M')f|H_{\sigma(k)}=\chi(M')T_nf.\end{aligned}$$

The form T_nf thus has the character $M\mapsto\chi(M')$. $\square$

7.5. **Proposition.** *The following cusp forms are Hecke eigenforms*:

$$
\begin{aligned}
&F_i := f_i, \qquad i = 1, 5, 6, 7, \\
&F_2 := f_2 - 4f_2' = \theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right] + 4\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right], \\
&F_3 := f_3 + 16f_3' = \theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right] + 16\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \\
&F_4 := f_4 + 4f_4' = \theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] + 4\theta\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right], \\
&g_i, \qquad i = 1, 2, 3, 4.
\end{aligned}
$$

Proof. Recall that the f_i and the g_i are determined by their character (cf. Theorem 6.4). Since both $\Gamma(2)$ and $\Gamma(2)'$ satisfy the conditions of Proposition 7.4, we have that $T_n f_i$ and $T_n g_i$ are (up to a scalar multiple) determined by a character.

An explicit computation shows that $T_{p^i} f$ has the same character if $p^i \equiv 1 \bmod 4$, and it has character $\overline{\chi_f}$, the complex conjugate of χ_f, if $p^i \equiv 3 \bmod 4$, where in fact $f \in S_3(\Gamma(2, 4, 8))$ can be any cusp form determined with a character.

The space spanned by such a cusp form f and its translates by the Hecke action is thus at most 2-dimensional. In particular, if there is no cusp form with character $\overline{\chi_f}$ or if χ_f is real-valued, then f is an eigenform. Using a computer, one then finds the eigenforms listed. □

7.6. In the table below we list the coefficients a_p, a_{p^2} of the Hecke polynomials corresponding to these eigenforms.

coef.	F_1	F_2	F_3	F_4	F_5	F_6	F_7	g_1	g_2	g_3	g_4
a_3	8	8	16	8	0	0	0	0	0	0	8
a_{3^2}	6	6	102	54	54	54	6	−18	6	30	6
a_5	28	28	28	20	12	−12	4	0	0	16	−32
a_{5^2}	190	190	190	350	30	30	−130	70	−10	230	310
a_7	80	80	32	−16	0	0	0	0	0	0	−32
a_{7^2}	2030	2030	−658	686	686	686	238	686	−18	−210	−658
a_{11}	88	88	176	−40	0	0	0	0	0	0	88
a_{11^2}	−3146	−3146	8470	2662	2662	2662	6	1694	−330	462	−3146
a_{13}	204	204	204	−28	60	−60	84	0	0	−80	−160
a_{13^2}	8398	8398		494		2290	4238	3094	−442	3510	390
a_{17}	356	356	356	4	−60	−60	36	−180	−92	20	356
a_{17^2}	25126	25126		8806		6630	−3162	15878	9894	−6970	25126
a_{19}	424	424	336	40	0	0	0	0	0	0	424
a_{19^2}	30438	30438	−3002	13718	13718	13718	10982	−12274	−8474	9918	30438

8. The Andrianov L-functions

In the table below we list the Fourier coefficients of some elliptic modular new forms which appear to be related to the Siegel cusp forms listed above.

form	space	a_3	a_5	a_7	a_{11}	a_{13}	a_{17}	a_{19}
ϕ_1	$S_2(\Gamma_0(32))$	0	-2	0	0	6	2	0
ψ_1	$S_3(\Gamma_0(32, (\frac{-1}{\cdot})))$	$4i$	2	$-8i$	$-4i$	-14	18	$12i$
ρ_1	$S_4(\Gamma_0(8))$	-4	-2	24	-44	22	50	44
ρ_2	$S_4(\Gamma_0(32))$	0	22	0	0	-18	-94	0
ρ_3	$S_4(\Gamma_0(32))$	8	-10	16	-40	-50	-30	40

8.1. The cusp form F_1 was studied in [6], where it was proven that F_1 is the Saito-Kurokawa lift of the elliptic modular form $\rho_1 \in S_4(\Gamma_0(8))$. Therefore,

$$L(F_1, s) = \zeta_{\mathbf{Q}}(s-1)\zeta_{\mathbf{Q}}(s-2)L(\rho_1, s).$$

(One easily checks that indeed $H_p(X) = (X-p)(X-p^2)(X^2 - a_pX + p^3)$ with the a_p from ρ_1.)

8.2. The first Hecke polynomials of F_2 suggest that its L-function is the same as that of F_1:

$$L(F_2, s) \stackrel{?}{=} \zeta_{\mathbf{Q}}(s-1)\zeta_{\mathbf{Q}}(s-2)L(\rho_1, s).$$

8.3. The L-function of F_3 also appears to be a twisted form of the L-function of F_1:

$$L(F_3, s) \stackrel{?}{=} \zeta_{\mathbf{Q}}(s-1)\zeta_{\mathbf{Q}}(s-2)L(\rho_1^{(3)}, s),$$

where the $^{(3)}$ stands for twisting at the primes 3 mod 4 (the L-function $L(\rho_1^{(3)}, s)$ is the L-function of a cusp form of weight 3 on $\Gamma_0(16)$).

8.4. The L-function of F_4 appears to be the product of two elliptic modular L-functions:

$$L(F_4, s) \stackrel{?}{=} L(\phi_1^{(-2)}, s-1)L(\rho_3^{(-2)}, s),$$

where the $^{(-2)}$ stands for twisting at primes $\equiv$ 5, 7 mod 8.

8.5. The modular form F_5 was also studied in [6], in fact it defines the holomorphic 3-form on the threefold Y studied there. Its L-series seems to be

$$L(F_5, s) \stackrel{?}{=} L(\phi_1, s-1)L(\rho_2, s).$$

8.6. The L-function of F_6 seems similar to the L-function of F_5:

$$L(F_6, s) \stackrel{?}{=} L(\phi_1^{(5)}, s-1)L(\rho_2^{(5)}, s),$$

where the $^{(5)}$ stands for twisting at the primes 5 mod 8 (note that one can also twist at the primes 3 mod 8 (or 7 mod 8) without changing the L-functions).

8.7. The L-function of F_7 seems to be related to the L-function of a Galois representation π of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is the tensor product of the Galois representations corresponding to ϕ_1 (a CM representation) and to ψ_1. At least for all primes ≤ 19 we have that the roots of the Hecke polynomial of F_7 are of the form $\alpha_i\beta_j$ with α_i the roots of the Hecke polynomial of ϕ_1 and β_i the roots of the Hecke polynomial ψ_1.

8.8. The form g_1 appears to be related to a Hecke character χ of the field $K = \mathbb{Q}(\zeta_8)$.

We define a Hecke character

$$\chi : A_K^* \longrightarrow \mathbb{C}^*, \qquad \chi(\dots, x_\wp, \dots) = \prod_\wp \chi_\wp(x_\wp),$$

with A_K^* the ideles of K and the product is taken over all places of K. The character will be unramified outside the prime over 2, which we will denote by ν. Since the class number of K is one and χ is trivial on K^* (embedded diagonally), it suffices to define only the infinite components and the component χ_ν at the prime over 2. In fact, it suffices to define only the restriction of χ_ν to $\mathcal{O}_\nu^*$. We give these data below.

As places at infinity we choose the complex embeddings $\sigma_i : K \hookrightarrow \mathbb{C}^*$,

$$\sigma_1 \colon\ \zeta_8 \mapsto e^{\frac{\pi i}{4}}, \qquad \text{and} \qquad \sigma_3 \colon\ \zeta_8 \mapsto e^{\frac{3\pi i}{4}}.$$

The infinity components of χ we then define by

$$\chi_{\infty,i} : \mathbb{C}^* \longrightarrow \mathbb{C}^*, \qquad \chi_{\infty,1}(z) := z^{-3}, \qquad \chi_{\infty,3}(z) := z^{-1}\overline{z}^{-2}.$$

As $\mathcal{O}_\nu^*/(1+\pi_\nu^4\mathcal{O}) = \mathcal{O}_\nu^*/(1+2\mathcal{O}) \cong (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ (where the first factor is generated by the image of ζ_8), the projection to the second factor will give a character, which is the restriction to $\mathcal{O}_\nu^*$ of the desired one:

$$\chi_\nu : \mathcal{O}_\nu^* \longrightarrow \{\pm 1\} \subset \mathbb{C}^*.$$

(With $\pi_\nu = 1 - \zeta_8$ as local parameter at ν, the subgroup generated by ζ_8 mod $(1+2\mathcal{O}_\nu)$ is just: $1,\ 1+\pi_\nu,\ 1+\pi_\nu^2,\ 1+\pi_\nu+\pi_\nu^3$, so χ_ν is trivial on these, and not trivial on the other four). Since any unit in $\mathcal{O}_K$ can be written as $u = \zeta^i(1-\sqrt{2})^j$, one has that $\chi_{\infty,1}(u)\chi_{\infty,3}(u)\chi_\nu(u) = 1$, and thus these data define indeed a Hecke character.

The L-function of χ is given by

$$L(\chi, s) := (2\pi)^{-1-2s}\Gamma(s)\Gamma(s-1)\prod_\wp{}' \left(1 - \chi_\wp(\pi_\wp)N\wp^{-s}\right)^{-1},$$

where the product is now over all primes except the ones at infinity and ν, which is ramified. To facilitate comparison with the Hecke polynomials, we define

$$H_{\chi,p}(X) := \prod_{\wp|p} \left(X^{e_\wp} - \chi_\wp(\pi_\wp)\right),$$

where $e_\wp := [\mathcal{O}_\wp/(\wp) : \mathbb{F}_p]$ is the degree of the residue field extension. Then the equality $L(g_1, s) = L(\chi, s)$ is equivalent to $H_p(X) = H_{\chi,p}(X)$ for all p.

8.9. To compute the $H_{\chi,p}$, we choose a generator $\pi_\wp$ in $\mathscr{O}_K$ for each of the primes $\wp$ over p. Then

$$\begin{aligned}\chi_\wp(\pi_\wp) &= \chi(1, 1, \dots, 1, \pi_\wp, 1, \dots) \\ &= \chi(\pi_\wp^{-1}, \pi_\wp^{-1}, \dots, \pi_\wp^{-1}, 1, \pi_\wp^{-1}, \dots) \\ &= \chi_{\infty,1}(\pi_\wp^{-1})\chi_{\infty,3}(\pi_\wp^{-1})\chi_\nu(\pi_\wp^{-1}),\end{aligned}$$

where the last step follows because χ is unramified outside ν.

In particular, if $p \equiv 7 \mod 8$, then there are two primes over p, and the generators $\pi_\wp$ and $\pi'_\wp$ of these prime ideals can be chosen to lie in $\mathbb{Z}[\sqrt{2}]$. Writing $\pi_\wp = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$, we have $a^2 - 2b^2 = p$, and thus a and b are odd. Since $\sqrt{2} = \pi_\nu^2 + \pi_\nu^3 \in \mathscr{O}_\nu/(2)$, we get $\pi_\wp = 1 + \pi_\nu^2 + \pi_\nu^3 \in \mathscr{O}_\nu^*/(1 + 2\mathscr{O}_\nu)$, and thus $\chi_\nu(\pi_\wp^{-1}) = -1$. For the infinite places one finds (with $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$) that $\chi_{\infty,1}(\pi_\nu^{-1}) = (a + b\sqrt{2})^3$ and $\chi_{\infty,3}(\pi_\nu^{-1}) = (a - b\sqrt{2})^3$. Therefore,

$$\chi(\pi_\wp) = \chi(\pi'_\wp) = -p^3, \qquad \text{and} \qquad H_{\chi,p} = (X^2 + p^3)^2.$$

If $p \equiv 5 \mod 8$, then the generators for the two primes over p can be chosen to lie in $\mathbb{Z}[i]$. Choosing a generator $\pi_\wp = a + bi$ with a odd and b even for such a prime, one finds that $\chi_\nu(\pi_\wp^{-1}) = +1$, $\chi_{\infty,1}(\pi_\wp^{-1}) = (a + bi)^3$ and $\chi_{\infty,3}(\pi_\wp^{-1}) = p(a + bi)$. Therefore,

$$\chi(\pi_\wp) = p(a + bi)^4, \qquad \text{and} \qquad H_{\chi,p} = (X^2 - p(a + bi)^4)(X^2 - p(a - bi)^4).$$

If $p \equiv 3 \mod 8$, then we choose the generators in $\mathbb{Z}[\sqrt{-2}]$ and let $\pi_\wp = a + b\sqrt{-2}$ be one of them. Since $a^2 + 2b^2 = p$, we must have a and b odd. Then $\pi_\wp = 1 + \pi_\nu^2 + \pi_\nu^3 + \cdots$ in $\mathscr{O}_\nu^*$ and thus $\chi_\nu(\pi_\wp^{-1}) = -1$. Furthermore, $\chi_{\infty,1}(\pi_\wp^{-1}) = (a + b\sqrt{-2})^3$ and $\chi_{\infty,3}(\pi_\wp^{-1}) = p(a - b\sqrt{-2})$. Therefore,

$$\chi(\pi_\wp) = -p^2(a + b\sqrt{-2})^2$$

and

$$H_{\chi,p} = (X^2 + p^2(a + b\sqrt{-2})^2)(X^2 + p^2(a - b\sqrt{-2})^2).$$

For $p \equiv 1 \mod 8$, there are four primes over p and the Hecke polynomial is not so easy to describe. However, one can check that indeed $H_{\chi,17} = H_{17}$.

8.10. We were not able to write the L-function of g_2 as a (product of) 'known' L-functions.

8.11. The Hecke polynomials of g_3 have similar properties to those of F_7, with ψ_1 replaced by a form in $S_3(\Gamma_0(2^?), (\frac{-2}{\cdot}))$.

8.12. The modular form g_4 also seems to be a (twisted) Saito-Kurokawa lift of the form ρ_1:

$$L(g_4, s) \stackrel{?}{=} L(\chi^{(-2)}, s-1)L(\chi^{(-2)}, s-2)L(\rho_1, s),$$

with $\chi^{(-2)} : (\mathbb{Z}/8\mathbb{Z})^* \to \{\pm 1\}$ the Dirichlet character with $\chi^{(-2)}(5) = \chi^{(-2)}(7) = -1$.

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Department of Mathematics, Rijksuniversiteit Utrecht, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands
E-mail address: geemen@math.ruu.nl

Universität Kaiserslautern, Fachbereich Mathematik, Universität Kaiserslautern, Erwin-Schrödinger-Str., 6750 Kaiserslautern, Germany
E-mail address: straten@mathematik.uni-kl.de